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Some Properties of Wigner Polynomials

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Abstract.

Such well-known scientists as Legendre, Gegenbauer, Jacobi, Lager and others were engaged in the study of various properties of orthogonal polynomials. They introduced the concept of various polynomials and determined their properties. With these polynomials the series in orthogonal polynomials are composed and the issue of representing a function with these series is studied; the convergence and summability of these series are also studied by various methods.

In the presented paper, the so-called Wigner polynomials are considered. These polynomials are involved in the definition of generalized spherical functions. Therefore, determining the properties of these polynomials would be useful for studying the convergence and summability of Fourier series with respect to generalized spherical functions.

In this paper, some basic properties of Wigner polynomials are studied. In particular, asymptotic formulas and some estimates for these polynomials are determined.

Keywords: asymptotic formula; integral representation; Jacobi polynomials; Legendre polynomials; Wigner polynomials.

Introduction

In the presented paper, theorems are proved that determines various properties of Wigner polynomials. In particular, an asymptotic representation and some estimates for these polynomials are obtained.

Main Part

Gelfand and Shapiro introduced the concept of generalized spherical functions [1] and connected with them functions $P_{mn}^{\ell}(x)$ that will be as

$$P_{mn}^{\ell}(x) = (-1)^{\ell-n} \cdot \frac{i^{m-n}}{2^{\ell}(m-n)!} \cdot \sqrt{\frac{(\ell-n)!(\ell+m)!}{(\ell+n)!(\ell-m)!}} \cdot (1-x)^{-\frac{m-n}{2}} \cdot (1+x)^{-\frac{m+n}{2}} \cdot \frac{d^{\ell-m}}{dx^{\ell-m}}$$

$$\left[(1-x)^{\ell-n}(1+x)^{\ell+n}\right].$$

The generalized spherical functions are sometimes called as Wigner's functions [2], so we call $P_{mn}^{\ell}(x)$ polynomials as Wigner polynomials.

It is known that [1]

$$P_{mn}^{\ell}(x) = P_{nm}^{\ell}(x) \tag{1}$$

and

$$P_{mn}^{\ell}(x) = P_{-m,-m}^{\ell}(x). \tag{2}$$

Due the relations (1) and (2), it is always possible to ensure that the inequalities $m - n \geq 0$ and $m + n \geq 0$ are satisfied.

Let's designate *by* $P_k^{(\alpha,\beta)}(x)$ the Jacobi polynomials; we have (see [3], p. 79)

$$P_k^{(\alpha,\beta)}(x) = \frac{(-1)^k}{2^k \cdot k!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^k}{dx^k}\left[(1-x)^{\alpha+k}(1+x)^{\beta+k}\right].$$

It is known that ([2], p.133)

$$P_k^{(\alpha,\beta)} = 2^m \cdot i^{n-m}\sqrt{\frac{(\ell-n)!(\ell+n)!}{(\ell-m)!(\ell+m)!}} \cdot (1-x)^{-\frac{m-n}{2}} \cdot (1+x)^{-\frac{m+n}{2}} \cdot P_{mn}^{\ell}(x), \tag{3}$$

where

$$\ell = k + \frac{\alpha+\beta}{2}, \quad m = \frac{\alpha+\beta}{2}, \quad n = \frac{\beta-\alpha}{2}.$$

When $m = n$ we have $\alpha = 0$, $\beta = 2m$, $k = \ell - m$ and

$$P_{\ell-m}^{(0,2m)}(x) = 2^m(1+x)^{-m}P_{mm}^{\ell}(x).$$

Hence, we obtain

$$P_{mm}^{\ell}(x) = \left(\frac{1+x}{2}\right)^m P_{\ell-m}^{(0,2m)}, \quad \ell \geq m.$$

Conclusion

For Jacobi polynomials we have ([3], p. 204)

$$P_k^{(\alpha,\beta)}(\cos\vartheta) = k^{-\frac{1}{2}}\pi^{-\frac{1}{2}}\left(\sin\frac{\vartheta}{2}\right)^{-\alpha-\frac{1}{2}}\left(\cos\frac{\vartheta}{2}\right)^{-\beta-\frac{1}{2}}\cdot\cos\left[\left(k+\frac{\alpha+\beta+1}{2}\right)\vartheta-\right.$$

$$\left.-\left(\alpha+\frac{1}{2}\right)\frac{\pi}{2}\right]+O(k^{-\frac{1}{2}}), \qquad (4)$$

where $0 < \vartheta < \pi$, and the estimate of the remainder term is uniform on the segment $[\varepsilon; \pi-\varepsilon]$ for a fixed $\varepsilon > 0$.

From (3) and (4) we obtain

$$2^m\cdot i^{n-m}\sqrt{\frac{(\ell-n)!(\ell+n)!}{(\ell-m)!(\ell+m)!}}\cdot(1-x)^{\frac{n-m}{2}}\cdot(1+x)^{-\frac{n+m}{2}}\cdot P_{mn}^{\ell}(x)=$$

$$=(\ell-m)^{-\frac{1}{2}}\ \pi^{-\frac{1}{2}}\left(\sin\frac{\vartheta}{2}\right)^{n-m-\frac{1}{2}}\left(\cos\frac{\vartheta}{2}\right)^{-m-n-\frac{1}{2}}\cdot\cos\left[\left(\ell+\frac{1}{2}\right)\vartheta-\right.$$

$$\left.-\left(m-n+\frac{1}{2}\right)\frac{\pi}{2}\right]+O\left[(\ell-m)^{-\frac{3}{2}}\right], \quad (5)$$

where $x = \cos\vartheta$.

From (5) follows

$$P_{mn}^{\ell}(\cos\vartheta)=\sqrt{\frac{(\ell-m)!(\ell+m)!}{(\ell-n)!(\ell+n)!}}\cdot\cdot\cos\left[\left(\ell+\frac{1}{2}\right)\vartheta-\left(m-n+\frac{1}{2}\right)\frac{\pi}{2}\right]+$$

$$+O\left[(\ell-m)^{-\frac{3}{2}}\right], 0 < \vartheta < \pi.$$

In particular, if m = n, we have

$$P_{mm}^{\ell}(\cos\vartheta)=\frac{\sqrt{2}\cos\left[\left(\ell+\frac{1}{2}\right)\vartheta-\frac{\pi}{4}\right]}{\sqrt{(\ell-m)\cdot\pi\cdot\sin\vartheta}}+O\left[(\ell-m)^{-\frac{3}{2}}\right], \quad 0 < \vartheta < \pi, \qquad (6)$$

At this estimate of the remainder term is uniform on the segment $[\varepsilon; \pi-\varepsilon]$.

Therefore, we have proved the validity of the theorem.

Theorem 1. If $0 < \vartheta < \pi$ and $\ell \geq m$, then for the polynomials $P_{mm}^{\ell}(\cos\vartheta)$ occurs an asymptotic representation

$$P_{mm}^{\ell}(\cos\vartheta)=\frac{\sqrt{2}\cos\left[\left(\ell+\frac{1}{2}\right)\vartheta-\frac{\pi}{4}\right]}{\sqrt{(\ell-m)\cdot\pi\cdot\sin\vartheta}}+O\left[(\ell-m)^{-\frac{3}{2}}\right],$$

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At this estimate of the remainder term is uniform on the segment $[\varepsilon;\ \pi-\varepsilon]$.

From (6) follows that

$$\left|P^{\ell}_{mm}(\cos\vartheta)\right| \le \frac{c}{\sqrt{(\ell-m)\pi\cdot\sin\vartheta}},\ 0<\vartheta<\pi$$

and

$$\left|P^{\ell}_{mm}(\cos\vartheta)\right| < \frac{c}{\sqrt{(\ell-m)}},\ \ \delta<\vartheta<\pi-\delta,$$

where $\delta>0$ c is constant.

Let's now obtain an estimation for the function $P^{\ell}_{mm}(\cos\vartheta)$ on the segment $[0;\ \pi]$.

Theorem 2. If $0\le\vartheta\le\pi$ and $\ell\ge m$, then

$$\left|P_{|m|,|m|}(\cos\vartheta)\right| \le \frac{1}{2}(1+2^{|m|}).$$

Proof. For the function $P^{\ell}_{mn}(x)$ occurs the formula of integral presentation ([2], p. 129)

$$P^{\ell}_{mn}(x) = \frac{1}{2\pi}\sqrt{\frac{(\ell-m)!(\ell+m)!}{(\ell-n)!(\ell+n)!}}\int_0^{2\pi}\left(\cos\frac{\vartheta}{2}\,e^{\frac{i\varphi}{2}} + i\sin\frac{\vartheta}{2}e^{-\frac{i\varphi}{2}}\right)^{\ell-n}\times$$

$$\times\left(i\sin\frac{\vartheta}{2}\cdot e^{\frac{i\varphi}{2}} + \cos\frac{\vartheta}{2}e^{-\frac{i\varphi}{2}}\right)^{\ell+n}\cdot e^{im\varphi}d\varphi, \tag{7}$$

where $x=\cos\vartheta$.

From (7) we obtain

$$P^{\ell}_{mm}(\cos\vartheta) = \frac{1}{2\pi}\int_0^{2\pi}\left(\cos\frac{\vartheta}{2}\,e^{\frac{i\varphi}{2}} + i\sin\frac{\vartheta}{2}e^{-\frac{i\varphi}{2}}\right)^{\ell-m}\left(i\sin\frac{\vartheta}{2}\,e^{\frac{i\varphi}{2}} + \cos\frac{\vartheta}{2}e^{-\frac{i\varphi}{2}}\right)^{\ell+m} e^{im\varphi}d\varphi$$

Thus

$$\left|P_{mm}(\cos\vartheta)\right| \le \frac{1}{2\pi}\int_0^{2\pi}\left|\cos\frac{\vartheta}{2}\,e^{\frac{i\varphi}{2}} + i\sin\frac{\vartheta}{2}e^{-\frac{i\varphi}{2}}\right|^{\ell-m}\times$$

$$\times\left|i\sin\frac{\vartheta}{2}\cdot e^{\frac{i\varphi}{2}} + \cos\frac{\vartheta}{2}e^{-\frac{i\varphi}{2}}\right|^{\ell+m}d\varphi. \tag{8}$$

We have

$$\left|\cos\frac{\vartheta}{2}\cdot e^{\frac{i\varphi}{2}} + i\sin\frac{\vartheta}{2}e^{-\frac{i\varphi}{2}}\right| = \left|\cos\frac{\vartheta}{2}\cos\frac{\varphi}{2} + \sin\frac{\vartheta}{2}\sin\frac{\varphi}{2} + \right.$$

$$+i\left(\cos\frac{\vartheta}{2}\sin\frac{\varphi}{2}+\ \sin\frac{\vartheta}{2}\cos\frac{\varphi}{2}\right)=\left(\cos^2\frac{\vartheta}{2}\cos^2\frac{\varphi}{2}+\right.$$

$$+2\sin\frac{\vartheta}{2}\cos\frac{\vartheta}{2}\sin\frac{\varphi}{2}\cos\frac{\varphi}{2}+\sin^2\frac{\vartheta}{2}\sin^2\frac{\varphi}{2}+\ \cos^2\frac{\vartheta}{2}\sin^2\frac{\varphi}{2}+$$

$$\left.+2\sin\frac{\vartheta}{2}\cos\frac{\vartheta}{2}\sin\frac{\varphi}{2}\cos\frac{\varphi}{2}+\sin^2\frac{\vartheta}{2}\cdot\cos^2\frac{\varphi}{2}\right)^{\frac{1}{2}}=$$

$$=\left(\cos^2\frac{\vartheta}{2}+\sin^2\frac{\vartheta}{2}+\ 4\sin\frac{\vartheta}{2}\cos\frac{\vartheta}{2}\sin\frac{\varphi}{2}\cos\frac{\varphi}{2}\right)^{\frac{1}{2}}=$$

$$=(1+\sin\varphi\cdot\sin\vartheta)^{\frac{1}{2}}.$$

Similarly, would be shown that

$$\left|i\sin\frac{\vartheta}{2}e^{\frac{i\varphi}{2}}+\ \cos\frac{\vartheta}{2}e^{-\frac{i\varphi}{2}}\right|=\left(\cos^2\frac{\vartheta}{2}+\sin^2\frac{\varphi}{2}-\ 4\sin\frac{\vartheta}{2}\cos\frac{\vartheta}{2}\sin\frac{\varphi}{2}\cos\frac{\varphi}{2}\right)^{\frac{1}{2}}=$$

$$=(1-\sin\varphi\cdot\sin\vartheta)^{\frac{1}{2}}.$$

Consequently, from (8) we obtain

$$\left|P^{\ell}_{mm}(\cos\vartheta)\right|\le\frac{1}{2\pi}\int_0^{2\pi}(1+\sin\varphi\cdot\sin\vartheta)^{\frac{\ell-m}{2}}\cdot(1-\sin\varphi\cdot\sin\vartheta)^{\frac{\ell+m}{2}}d\varphi=$$

$$=\frac{1}{2\pi}\int_0^{\pi}(1+\sin\varphi\cdot\sin\vartheta)^{\frac{\ell-m}{2}}\cdot(1-\sin\varphi\cdot\sin\vartheta)^{\frac{\ell+m}{2}}d\varphi+$$

$$+\frac{1}{2\pi}\int_{\pi}^{2\pi}(1+\sin\varphi\cdot\sin\vartheta)^{\frac{\ell-m}{2}}\cdot(1-\sin\varphi\cdot\sin\vartheta)^{\frac{\ell+m}{2}}d\varphi=$$

$$=\frac{1}{2\pi}\int_0^{\pi}(1+\sin\varphi\cdot\sin\vartheta)^{\frac{\ell-m}{2}}(1-\sin\varphi\cdot\sin\vartheta)^{\frac{\ell+m}{2}}d\varphi+$$

$$+\frac{1}{2\pi}\int_0^{\pi}(1-\sin\varphi\cdot\sin\vartheta)^{\frac{\ell-m}{2}}(1+\sin\varphi\cdot\sin\vartheta)^{\frac{\ell+m}{2}}d\varphi.$$

As [1] $P^{\ell}_{mm}(1)=1$ and $P^{\ell}_{mm}(-1)=(-1)^{\ell}$, therefore, let's consider the case $0<\vartheta<\pi$. For such ϑ we have $1+\sin\varphi\cdot\sin\vartheta>1$ and $1-\sin\varphi\cdot\sin\vartheta<1$.

Let's $m>0$. Then $(1+\sin\varphi\cdot\sin\vartheta)^{\frac{\ell-m}{2}}<(1+\sin\varphi\cdot\sin\vartheta)^{\frac{\ell+m}{2}}$, therefore

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$$\left|P^{\ell}_{mm}(\cos\vartheta)\right| \le \frac{1}{2\pi}\int_0^{\pi}(1+\sin\varphi\cdot\sin\vartheta)^{\frac{\ell+m}{2}}\,(1-\sin\varphi\cdot\sin\vartheta)^{\frac{\ell+m}{2}}\,d\varphi+$$

$$+\frac{1}{2\pi}\int_0^{\pi}(1-\sin\varphi\cdot\sin\vartheta)^{\frac{\ell-m}{2}}\,(1+\sin\varphi\cdot\sin\vartheta)^{\frac{\ell-m}{2}}\,(1+\sin\varphi\cdot\sin\vartheta)^{\frac{\ell+m}{2}-\frac{\ell-m}{2}}\,d\varphi=$$

$$\int_0^{\pi}(1-\sin^2\varphi\cdot\sin^2\vartheta)^{\frac{\ell+m}{2}}\,d\varphi+\frac{1}{2\pi}\int_0^{\pi}(1-\sin^2\varphi\cdot\sin^2\vartheta)^{\frac{\ell-m}{2}}\times$$

$$\times(1+\sin\varphi\cdot\sin\vartheta)^m\,d\varphi \le \frac{1}{\pi}\int_0^{\pi}d\varphi+\frac{1}{2\pi}\cdot 2^m\int_0^{\pi}d\varphi=\frac{1}{2}+\frac{1}{2}\cdot 2^m=\frac{1}{2}(1+2^m)\,.$$

If $m < 0$, we have

$$\left|P_{m,m}(\cos\vartheta)\right| = \left|P^{\ell}_{-m,-m}(\cos\vartheta)\right| \le \frac{1}{2}(1+2^{-m})\,.$$

It is clear that if $m = 0$, then

$$\left|P^{\ell}_{0,0}(\cos\vartheta)\right| = \left|P_{\ell}(\cos\vartheta)\right| \le 1\,,$$

where P_{ℓ} – is the Legendre polynomial.

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